The effective range theory

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Objective:

To obtain the energy dependence of low energy scattering by introduction of another parameter called as effective range.

Since we are dealing with low energy so it is safe to assume $l = 0$, hence the equations can be written as;

Set 1 - With potential $V(r)$

$$u''(r) + \{k^2 - U(r)\} u(r) = 0 \longrightarrow \text{Low energy scattering} \quad (1)$$

$$u_0''(r) - U(r) u(r) = 0 \longrightarrow \text{Limiting case of zero-energy scattering.} \quad (2)$$

the suffix o refers to the limiting case of zero energy

Sol:- without potential

$$v''(r) + k^2 v(r) = 0 \quad (3)$$

$$v_0''(r) = 0 \quad (4)$$

where $k^2 = \frac{2\mu E}{\hbar^2} = \frac{ME}{\hbar^2}$

$$U(r) = \frac{M V(r)}{\hbar^2}$$

Multiplying eqn ① by u_0 and eqn ② by u and subtracting the latter from the former

$$\left.\begin{aligned} u''u_0 + uu_0(K^2 - U) &= 0 \\ u_0''u - Uuu_0 &= 0 \end{aligned}\right\} \Rightarrow u_0''u - u''u_0 = K^2uu_0$$

$$\boxed{\frac{d}{dr}(u_0'u - u'u_0) = K^2uu_0} \qquad (5)$$

Similarly we can write

$$\boxed{\frac{d}{dr}(v_0'v - v'v_0) = K^2vv_0} \qquad (6)$$

Now subtracting ⑥ from ⑤ and we get

$$\frac{d}{dr}\left(u_0' u - u' u_0 - v_0' v + v' v_0\right) = k^2 (u u_0 - v v_0) \quad \text{—— (7)}$$

Integrating eqn (7) with limits $r = 0$ to $r = \infty$ we get

$$\left(u_0' u - u' u_0 - v_0' v + v' v_0\right)_{r=0}^{r=\infty} = k^2 \int_0^\infty (u u_0 - v v_0)\, dr$$

Using following assumption

(1) for outside the potential region

$$u(r) = v(r) \quad \text{and} \quad u_0(r) = v_0(r)$$

(2) $u(r) = u_0(r) = 0$ at $r = 0 \Rightarrow u(0) = u_0(0) = 0$

$v(r) = v_0(r) = 1$ at $r = 0 \Rightarrow v(0) = v_0(0) = 1$

Using eq^n ④

$$v_0''(r) = 0$$

$$v_0(r) = D(r-a)$$

since $v_0(0) = 1$ at $r = 0$

$$1 = -Da \Rightarrow D = -\frac{1}{a}$$

So $v_0(r) = -\frac{1}{a}(r-a) = 1 - \frac{r}{a}$

$$v_0'(r) = -\frac{1}{a}$$

$$v_0'(0) = -\frac{1}{a}$$

Outside potential region

$$u(r) = v(r) = A \frac{\sin(kr + \delta)}{k}$$

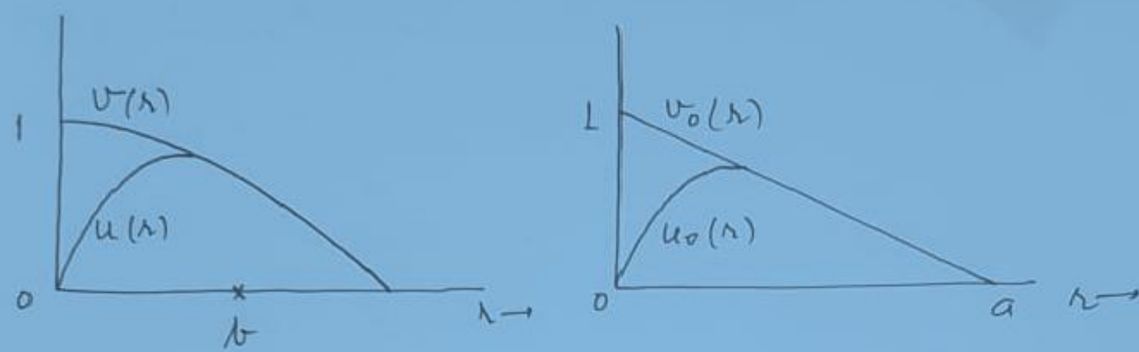
at $r = 0$ $\quad v(r=0) = v(0) = 1$

$$A = \frac{k}{\sin\delta}$$

$$v(r) = \frac{\sin(kr+\delta)}{\sin\delta}$$

$$v'(r) = \frac{k\cos(kr+\delta)}{\sin\delta}$$

$$\boxed{v'(0) = k\cot\delta}$$



Radial wave functions for effective range theory

$$v(0) = 1, \quad v_0(0) = L$$

$$v_0'(0) = -\frac{1}{a}, \quad v'(0) = K \cot \delta$$

substituting these rusits in the following eqn

$$\left(v_0' v - v' v_0 \right)_{r=0} = K^2 \int_0^\infty (u u_0 - v v_0)\, dr$$

$$\Rightarrow -\frac{1}{a} - k\cot\delta_0 = k^2 \int_0^\infty (uu_0 - vv_0)\,dr$$

Since

Outside potential region

$$u(r) = v(r) \quad \& \quad u_0(r) = v_0(r)$$

Inside potential region, for low energy

$$u(r) = u_0(r) \quad \text{and} \quad v(r) = v_0(r)$$

for $E \ll V(r)$, now defining the effective range r_0

$$r_0 = 2\int_0^\infty (v_0^2 - u_0^2)\,dr$$

Although the integration

$$r_0 = 2\int_0^\infty (v_0^2 - u_0^2)\,dr$$

extends to ∞, but r_0 gets contribution effectively only from the potential region only.

$$-\frac{1}{a} - k\cot\delta = k^2\int (v_0^2 - u_0^2)\,dr = \frac{1}{2} r_0 k^2$$

$$\Rightarrow \boxed{\cot\delta = \frac{1}{k}\left(\frac{1}{2} r_0 k^2 - \frac{1}{a}\right)}$$

Since we know that

$$\sigma_{total}(E) = \frac{4\pi}{K^2}\sin^2\delta$$

$$= \frac{4\pi}{K^2 \operatorname{cosec}^2\delta}$$

$$\sigma_{total}(E) = \frac{4\pi}{K^2 + \left(\frac{1}{2}r_0 K^2 - \frac{1}{a}\right)^2}$$

$a \rightarrow$ fermi scattering length.

The parameters a and r_0 do not depend on the form and shape of the potential well.

$\therefore$ The effective range theory is also known as the shape independent theory.

Reference Book

- Nuclear Physics by Devnathan